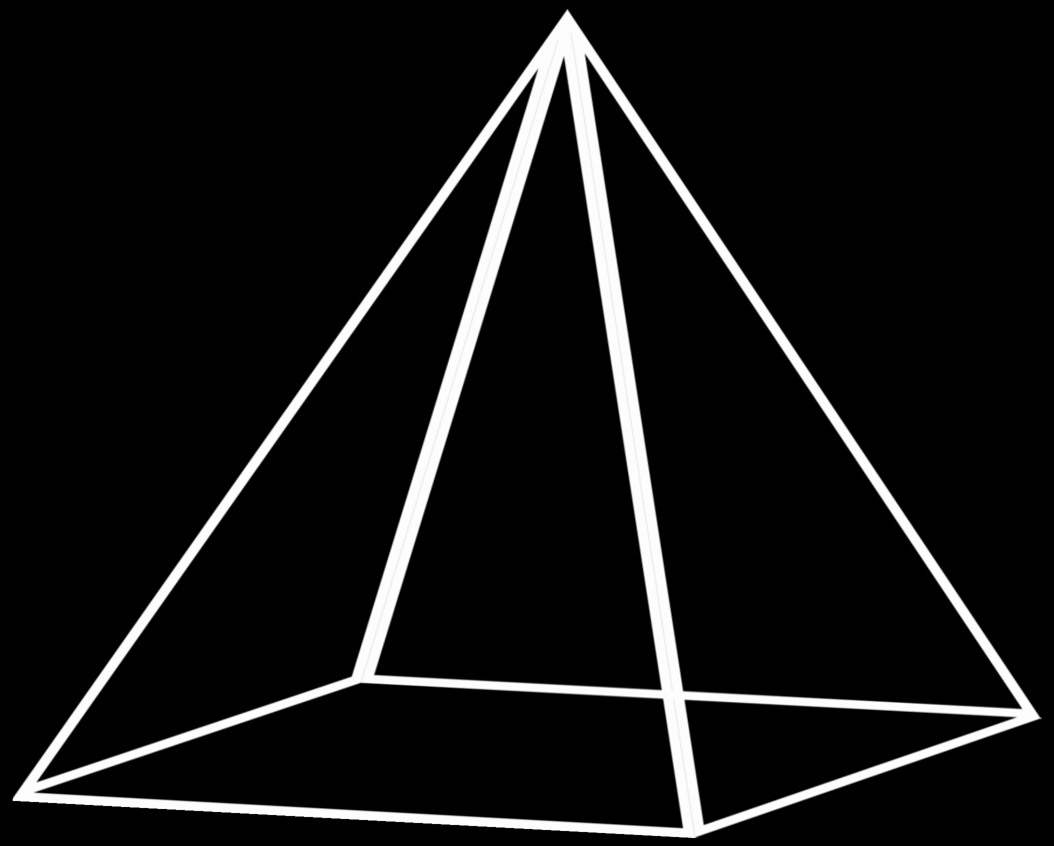
STEREOMETRIJA

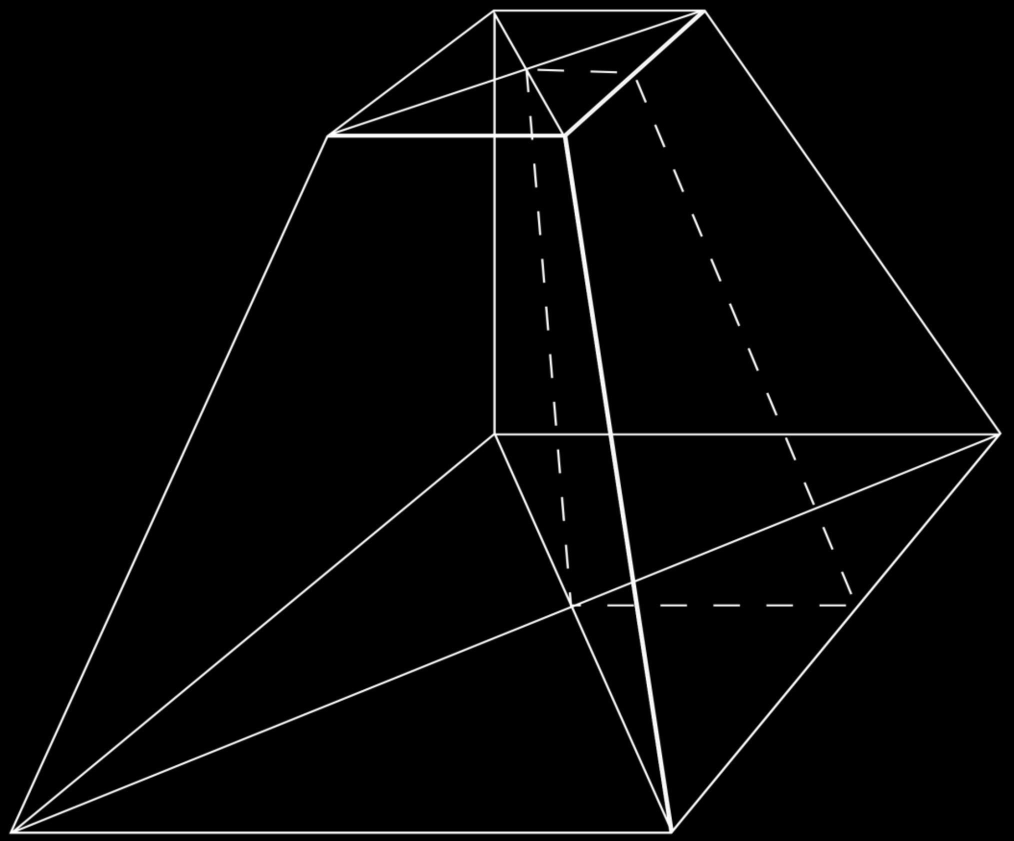
PIRAMIDA

- $P = B + M$
- $V = \frac{B * H}{3}$



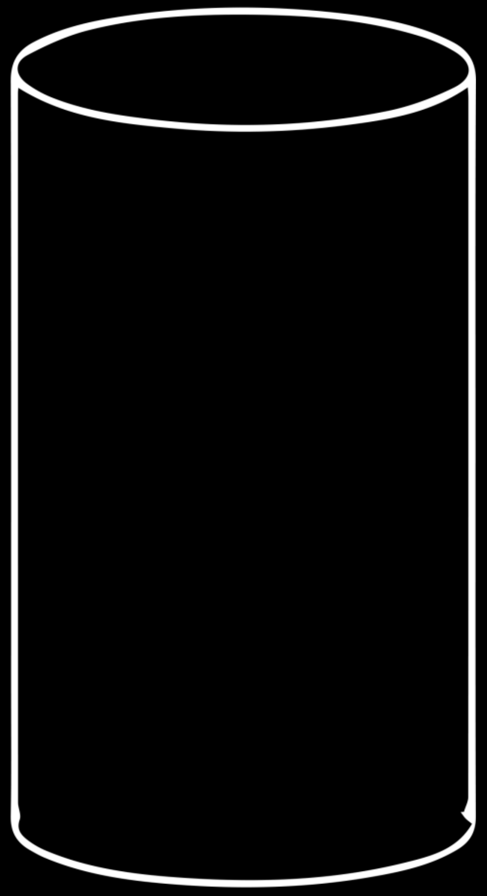
ZARUBLJENA PIRAMIDA

- $P = B_1 + B_2 + M$
- $V = \frac{(B_1 + B_2 + \sqrt{B_1 B_2}) * H}{3}$



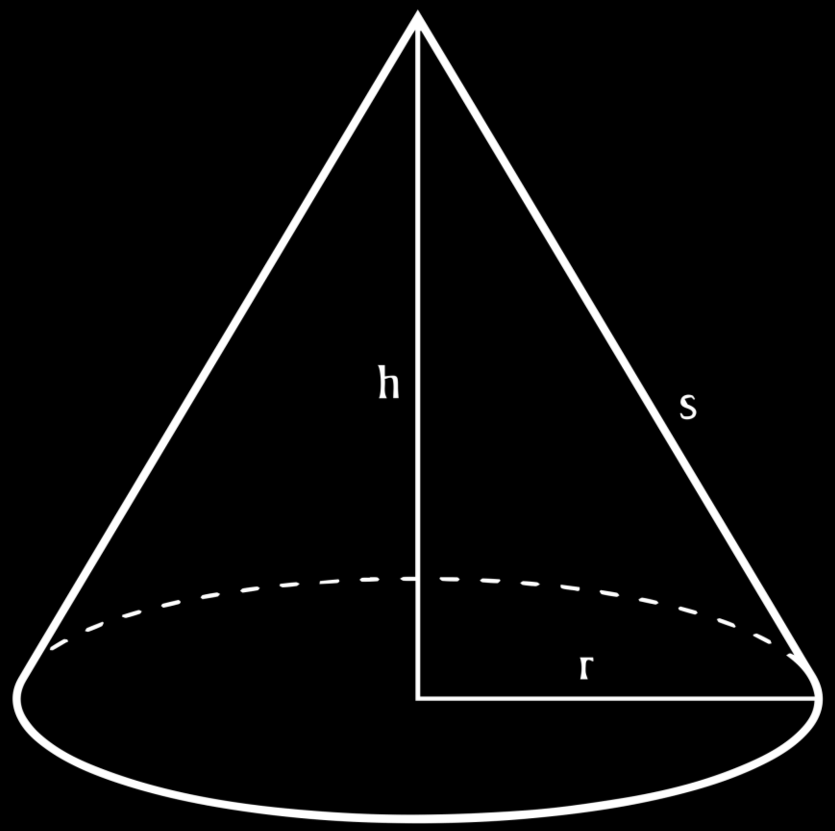
VALJAK

- $B = r^2 \pi$
- $M = 2\, r \pi * H$
- $P = 2B + M$
- $V = B * H$



KUPA

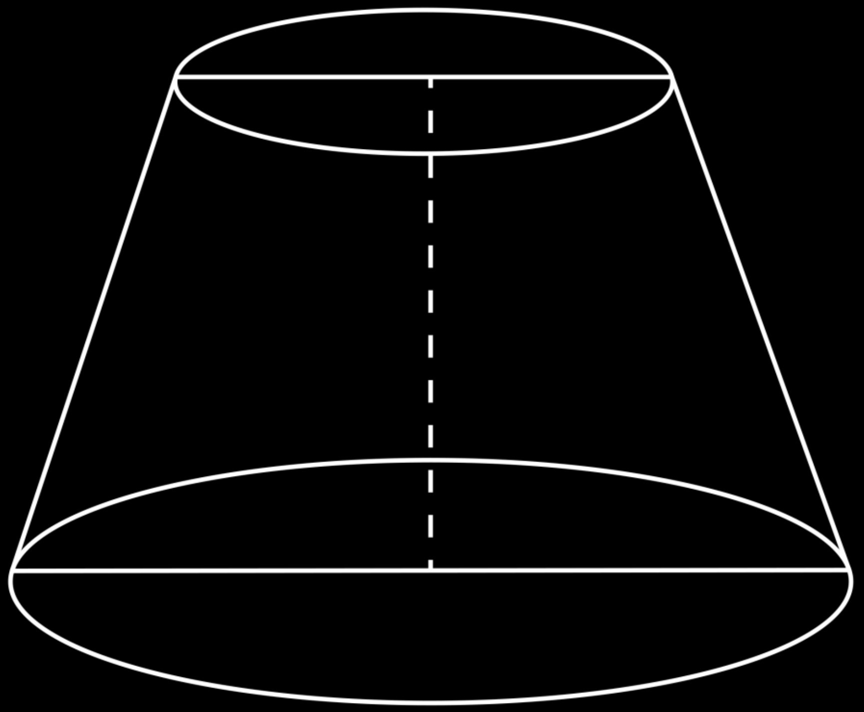
- $B = r^2\pi$
- $M = r$
- $P = M + B$
- $V = \frac{B * H}{3}$



ZARUBLJENA KUPA

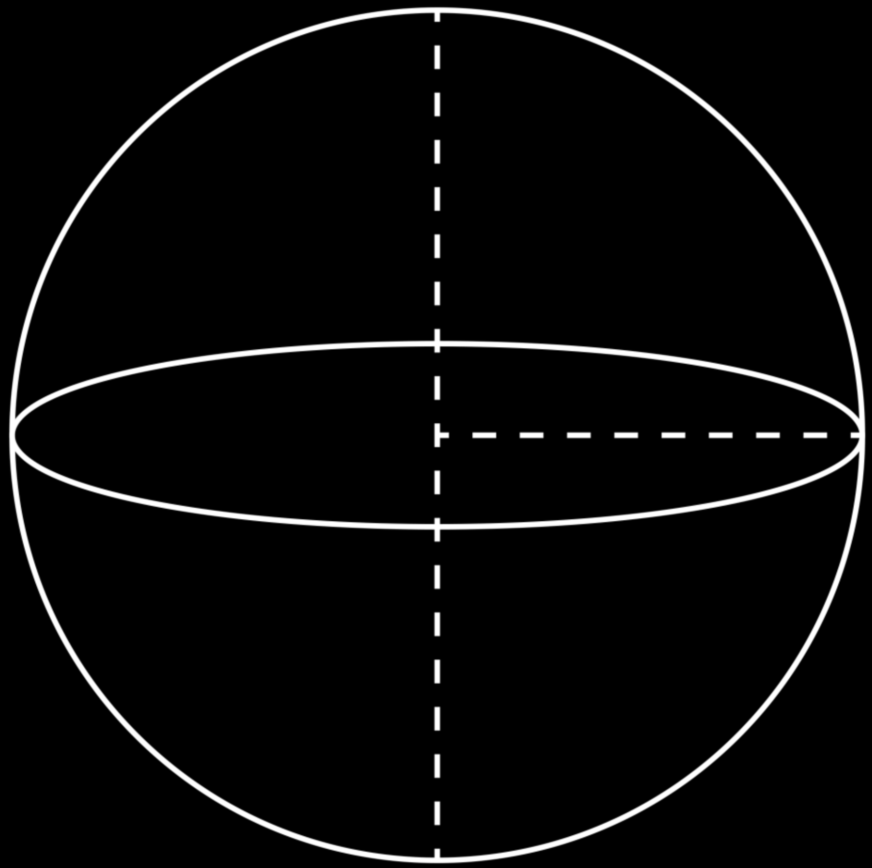
- $B_1 = R^2\pi$; $B_2 = r^2\pi$
- $M = s(R + r)\pi$
- $P = B_1 + B_2 + M$

$$V = \frac{H}{3}\left(B_1 + B_2 + \sqrt{B_1 B_2}\right)$$



LOPTA

- $P = 4\,R^2\pi$
- $V = \frac{4}{3}\,r^3\pi$



ZADACI

37. Bočna ivica pravilne trostrane piramide je 10, a stranica osnove 12. Odredi rastojanje od centra osnove piramide do sredine apoteme.

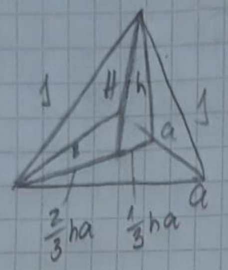
38. Visina i stranica osnove pravilne trostrane piramide jednake su i imaju dužine 34. Odredi rastojanje od centra osnove piramide do boćne ivice.

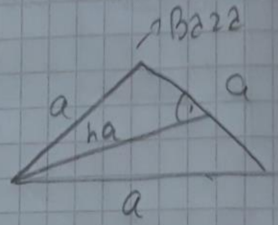
37.

$s = 10$

$a = 12$

$\frac{h}{2} = ?$





$h_a = \frac{\sqrt{3}}{2} a$

$h_a = 6\sqrt{3}$

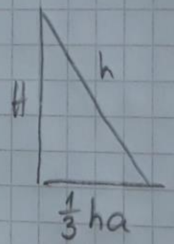
$\frac{2}{3} h_a = 4\sqrt{3}$

$\frac{1}{3} h_a = 2\sqrt{3}$

$H^2 = s^2 - \left(\frac{2}{3} h_a\right)^2$

$H^2 = 100 - 48$

$H = 2\sqrt{13}$



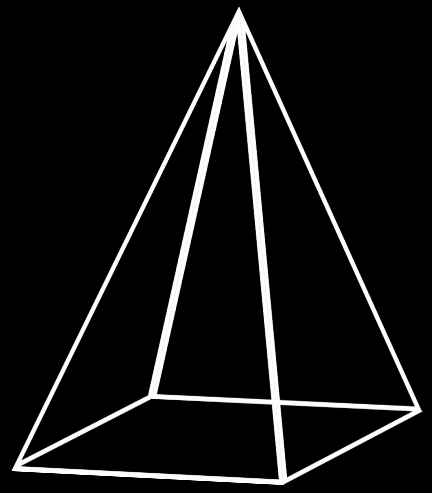
$h^2 = H^2 + 4 \cdot 3$

$h^2 = 64$

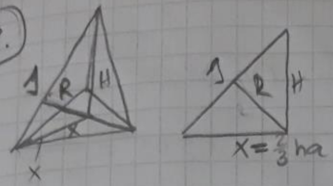
$h = 8$

$\frac{h}{2} = 4$

apotema = bočna visina



38.



$H = a = 34$

$h_a = \frac{a\sqrt{3}}{2} = 17\sqrt{3}$

$x = \frac{2}{3} h = \frac{34\sqrt{3}}{3}$

$s^2 = x^2 + H^2$

$s^2 = 1156 + \frac{1156 \cdot 3}{9}$

$s^2 = \frac{4624}{3}$ $\quad s = \frac{68}{\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{68\sqrt{3}}{3}$

$P_\Delta = \frac{H \cdot x}{2} = \frac{34 \cdot \frac{34\sqrt{3}}{3}}{2} = \frac{\frac{1156\sqrt{3}}{3}}{\frac{2}{1}} = \frac{1156\sqrt{3}}{6} = \frac{578\sqrt{3}}{3}$

$P_\Delta = \frac{s \cdot R}{2}$

$\frac{578\sqrt{3}}{3} = \frac{\frac{68\sqrt{3}}{3} \cdot R}{2} \Rightarrow \frac{1156\sqrt{3}}{3} = \frac{68\sqrt{3} \cdot R}{3}$

$1156\sqrt{3} = 68\sqrt{3} \cdot R$

$R = 17$

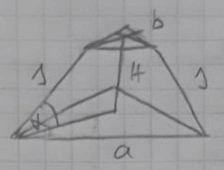
22. Bočne ivice pravilne trostrane zarubljene piramide nagnute su prema ravni veće osnove pod uglom α. Ivica veće osnove je a, a manje osnove b. Kolika je zapremina zarubljene piramide?

24. Data je površina $B_1 = 36$ veće osnove zarubljene piramide, njena zapremina $V = 104$ i visina $H = 6$. Kolika je zapremina dopunske piramide?

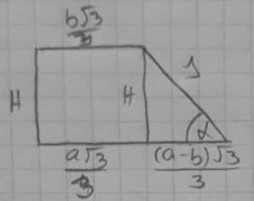
(22)

α, a, b

$V = ?$



$V = \frac{(B_1 + B_2 + \sqrt{B_1 B_2}) \cdot H}{3}$



$B_1 = \frac{a^2\sqrt{3}}{4}$

$B_2 = \frac{b^2\sqrt{3}}{4}$

$tg\,\alpha = \frac{H}{\frac{(a-b)\sqrt{3}}{3}}$

$H = \frac{(a-b)\sqrt{3}}{3} \cdot tg\,\alpha$

$V = \frac{(B_1 + B_2 + \sqrt{B_1 B_2}) \cdot H}{3}$

$V = \frac{\left(\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \sqrt{\frac{a^2\sqrt{3}}{4} \cdot \frac{b^2\sqrt{3}}{4}}\right) \cdot \frac{(a-b)\sqrt{3}}{3} tg\,\alpha}{3}$

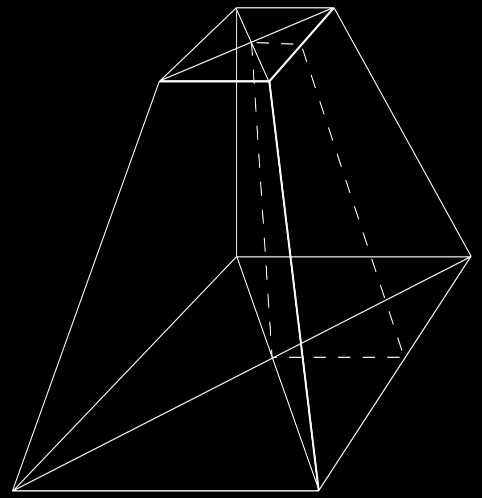
$V = \frac{\left(\frac{a^2\sqrt{3}}{4} + \frac{b^2\sqrt{3}}{4} + \sqrt{\frac{a^2\sqrt{3}}{4} \cdot \frac{b^2\sqrt{3}}{4}}\right) \cdot \frac{(a-b)\sqrt{3}}{3} \cdot tg\,\alpha}{3}$

$V = (a^2 + b^2 + ab)\frac{\sqrt{3}}{4} \cdot \frac{(a-b)\sqrt{3}}{9} tg\,\alpha$

$V = (a^2 + b^2 + ab)\frac{(a-b)}{12} tg\,\alpha$

$V = \frac{a^3 + ab^2 + a^2b - ba^2 - b^3 - ab^2}{12} tg\,\alpha$

$V = \frac{a^3 - b^3}{12} tg\,\alpha$



(24.) – zarubljena piramida ne kaže koja

$B_1 = 36$

$V = 104$

$H = 6$

$V = \frac{H}{3}(B_1 + B_2 + \sqrt{B_1 B_2})$

$104 = \frac{6}{3}(36 + B_2 + \sqrt{36 \cdot B_2})$

$104 = 2(36 + B_2 + \sqrt{36 \cdot B_2})$ $\qquad \sqrt{B_2} = t$

$t^2 + 6t + 36 = 52$

$t^2 + 6t - 16 = 0$

$t_{1/2} = \frac{-6 \pm \sqrt{36+64}}{2}$ $\quad t_1 = -8$ – ne može negativan $\quad t_2 = 2$

$\sqrt{B_2} = 2$

$B_2 = 4$

$\frac{B_1}{B_2} = \frac{(H+h)^2}{h^2} \;/\sqrt{\ }$

$\frac{\sqrt{B_1}}{\sqrt{B_2}} = \frac{H+h}{h} \;/\cdot h$

$\frac{\sqrt{B_1} \cdot h}{\sqrt{B_2}} = H + h \;/\cdot \sqrt{B_2}$

$\sqrt{B_1} \cdot h = \sqrt{B_2} \cdot H + \sqrt{B_2}\, h$

$\sqrt{B_1} \cdot h - \sqrt{B_2} \cdot h = \sqrt{B_2} \cdot H$

$h(\sqrt{B_1} - \sqrt{B_2}) = \sqrt{B_2} \cdot H$

$h = \frac{\sqrt{B_2} \cdot H}{\sqrt{B_1} - \sqrt{B_2}}$

$h = 3$

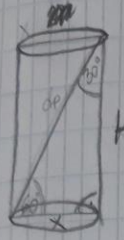
$V = \frac{B_2 \cdot h}{3}$

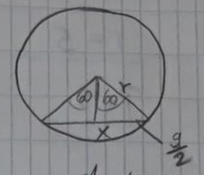
$V_1 = \frac{8}{3}$

24. Dijagonala presjeka valjka, sa ravni paralelnoj osi valjka, jednaka je 9 i nagnuta je prema ravni osnove pod uglom od 60°. Odredi površinu valjka ako je u osnovi valjka odsječen luk od 120°.

25. Ravan siječe osnove valjka po tetivama dužine 6 i 8, između kojih je rastojanje 9. Odredi površinu valjka ako je poluprečnik osnove 5 i ravan presijeca osu valjka (u unutrašnjoj tački valjka).

24.





$d_p = 9,\ 60°$

$-120°$

$x = \frac{1}{2} d_p$

$x = \frac{9}{2}$

$H^2 = d_p^2 - x^2$

$H^2 = 81 - \frac{81}{4}$

$H^2 = \frac{243}{4}$

$H = \frac{\sqrt{243}}{2} = \frac{3}{2}\sqrt{27} = \frac{9}{2}\sqrt{3}$

$H = \frac{9}{2}\sqrt{3}$

$P = \frac{54}{4}\pi + \frac{81}{2}\pi$

$P = \frac{54}{4}\pi + \frac{162}{4}\pi$

$P = 54\pi$!!!

Rešenje

$\sin 60° = \frac{\frac{x}{2}}{r}$

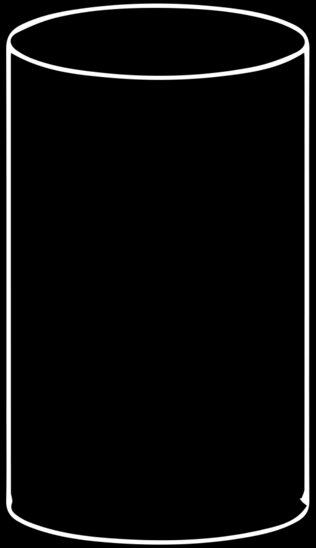
$r = \frac{\frac{x}{2}}{\sin 60°} = \frac{\frac{9}{4}}{\frac{\sqrt{3}}{2}} = \frac{\frac{9}{2}}{\frac{\sqrt{3}}{1}} = \frac{9}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}} = \frac{9\sqrt{3}}{6} = \frac{3\sqrt{3}}{2}$

$r = \frac{3\sqrt{3}}{2}$

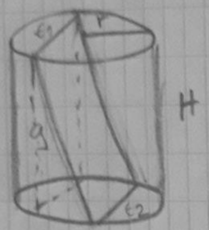
$B = r^2\pi$

$M = 2r\pi \cdot H$

$P = 2B + M \quad P = \frac{54}{4}\pi + 3\sqrt{3}\pi \cdot \frac{9\sqrt{3}}{2}$



25.



$t_1 = 6$

$t_2 = 8$

$t_1 \to t_2 = 9$

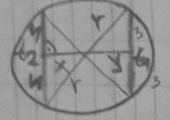
$r = 5$

$P = ?$

$B = r^2\pi$

$M = 2r\pi \cdot H$

$H = ?$



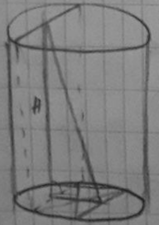
$r^2 = x^2 + 4^2$

$x^2 = 25 - 16$

$x = 3$

$y^2 = r^2 - 9$

$y^2 = 4$



$9^2 = H^2 + (x+y)^2$

$H^2 = 81 - 49$

$H = \sqrt{32}$

$H = 2\sqrt{8}$

$H = 4\sqrt{2}$

$B = 25\pi$

$M = 2r\pi \cdot H$

$M = 10\pi \cdot 4\sqrt{2}$

$P = 2B + M$

$P = 50\pi + 40\sqrt{2}\pi$

41. Trougao, čija je jednaka stranica 7, razlika preostale dvije stranice 5 i poluprečnik opisane kružnice $\frac{7\sqrt{3}}{3}$ rotira oko najduže stranice. Izračunaj površinu i zapreminu nastalog tijela.

43. Dat je trougao ABC sa tupim uglom α, uglom β i stranicom c $(=AB)$. Odredi zapreminu tijela koje nastaje rotacijom trougla ABC oko stranice c.

n1.

$a = 7$

$b - c = 5$

$R = \frac{7\sqrt{3}}{3}$

$$\frac{a}{\sin\alpha} = \frac{b}{\sin\beta} = \frac{c}{\sin\gamma} = 2R$$

$$\frac{7}{\sin\alpha} = \frac{2\cdot 7\sqrt{3}}{3}$$

$2\sqrt{3}\sin\alpha = 3$

$\sin\alpha = \frac{3}{2\sqrt{3}} \cdot \frac{\sqrt{3}}{\sqrt{3}}$

$\sin\alpha = \frac{\sqrt{3}}{2}$

$\alpha = 60°$

$b - c = 5$

$b = c + 5$

$a^2 = b^2 + c^2 - 2bc\cdot\cos\alpha$

$49 = c^2 + 10c + 25 + c^2 - (c+5)\cdot c$

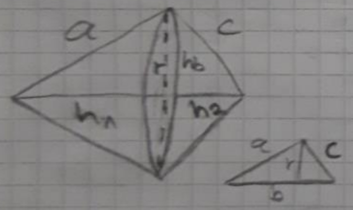
$49 = 2c^2 + 10c + 25 - (c+5)c$

$49 = 2c^2 + 10c + 25 - c^2 - 5c$

$c^2 + 5c - 24 = 0$

$c_1 = 3$ $\quad$ ~~$c_2 = -8$~~

$b = 3 + 5 = 8$



$h_1 + h_2 = b$ $\quad$ $B = h_b^2\pi$

$V = V_1 + V_2$

$V = \frac{B\cdot h_1}{3} + \frac{B h_2}{3}$

$V = \frac{B(h_1 + h_2)}{3}$

$V = 18\pi$

$s = \frac{a+b+c}{2} = 9$

$P = \sqrt{s(s-a)(s-b)(s-c)}$

$P = 6\sqrt{3}$

$P = \frac{h_b \cdot b}{2} \Rightarrow h_b = \frac{2P}{b}$

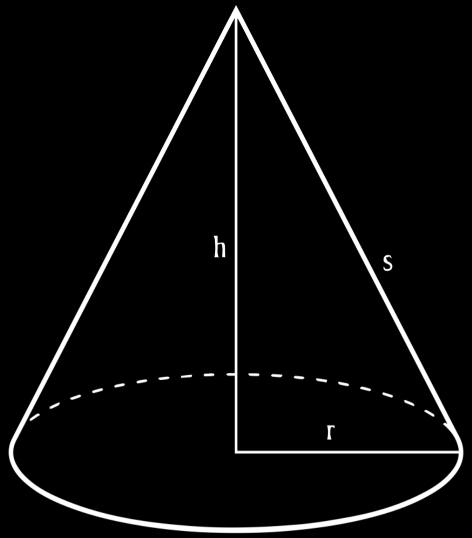
$h_b = \frac{12\sqrt{3}}{8}$

$h_b = \frac{3\sqrt{3}}{2}$

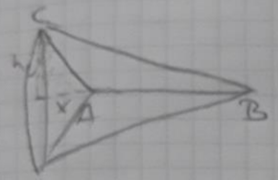
$P = M_1 + M_2$

$P = r\pi a + r\pi c$

$P = 15\sqrt{3}\pi$



13. $\alpha, \beta, c = AB$



$V = ?$

$V = V_1 - V_2$

1: $r = h_c$, $H = c + x$

2: $r = h_c$, $H = x$

$V = V_1 - V_2$

$$V = \frac{r^2\pi}{3}(c + x) - \frac{r^2\pi x}{3}$$

$$V = \frac{r^2\pi}{3}(c + x - x)$$

$$V = \frac{r^2\pi}{3}\cdot c = \frac{c^3\sin^2\alpha\sin^2\beta}{3\sin^2(\alpha+\beta)}\pi$$

$\alpha + \beta + \gamma = 180°$

$\gamma = 180° - (\alpha + \beta)$

$$\frac{a}{\sin\alpha} = \frac{c}{\sin\gamma}$$

$$a = \frac{c\cdot\sin\alpha}{\sin\gamma} = \frac{c\cdot\sin\alpha}{\sin(180° - (\alpha+\beta))} = \frac{c\sin\alpha}{\sin(\alpha+\beta)} = a$$

$$\frac{b}{\sin\beta} = \frac{c}{\sin\gamma}$$

$$b = \frac{c\sin\beta}{\sin\gamma} = \frac{c\cdot\sin\beta}{\sin(\alpha+\beta)} = b$$

$$P = \frac{ab\sin\gamma}{2} = \frac{c h_c}{2}$$

$$h = \frac{ab\sin\gamma}{c} = \frac{c\sin\alpha\cdot c\cdot\sin\beta\cdot\sin(\alpha+\beta)}{\sin(\alpha+\beta)\,(\sin(\alpha+\beta))\,c}$$

$$h_c = \frac{c\sin\alpha\sin\beta}{\sin(\alpha+\beta)}$$

$x^2 = b^2 - h_c^2$

$$x^2 = \frac{c^2\sin^2\beta}{\sin^2(\alpha+\beta)} - \frac{c^2\sin^2\alpha\sin^2\beta}{\sin^2(\alpha+\beta)}$$

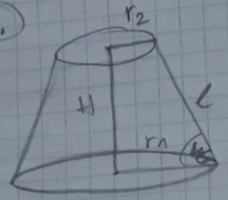
$$x^2 = \frac{c^2\sin^2\beta(1 - \sin^2\alpha)}{\sin^2(\alpha+\beta)}$$

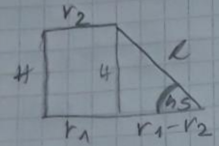
$$x^2 = \frac{c^2\sin^2\beta\cos^2\alpha}{\sin^2(\alpha+\beta)} \Rightarrow x = \frac{c\sin\beta\cos\alpha}{\sin(\alpha+\beta)}$$

22. Visina zarubljene kupe je H, a poluprečnici osnova odnose se kao 3 : 1. Odredi površinu i zapreminu zarubljene kupe ako je izvodnica nagnuta prema ravni veće osnove pod uglom od 45°.

31. Poluprečnici osnova prave zarubljene kupe su R i r, a izvodnica l. Odredi izvodnicu i visinu kupe od koje je nastala data zarubljena kupa.

22.





H

$r_1 : r_2 = 3:1 \Rightarrow r_1 = 3r_2$

$P, V = ?$

$\measuredangle 45°$

$$tg\,45° = \frac{H}{r_1-r_2}$$

$$tg\,45° = \frac{H}{2r_2}$$

$$1 = \frac{H}{2r_2}$$

$$H = 2r_2$$

$$r_2 = \frac{H}{2}$$

$$r_1 = \frac{3H}{2}$$

$$l^2 = H^2 + (2r_2)^2$$

$$l^2 = H^2 + 4r_2^2$$

$$l^2 = H^2 + 4\cdot\left(\frac{H}{2}\right)^2$$

$$l^2 = H^2 + H^2$$

$$l^2 = 2H^2$$

$$l = H\sqrt{2}$$

$B_1 = r_1^2\pi$

$B_2 = r_2^2\pi$

$M = l\pi(r_1+r_2)$

$P = B_1 + B_2 + M$

$$P = \frac{9H^2}{4}\pi + \frac{H^2}{4}\pi + H\sqrt{2}\,\pi\left(\frac{3H}{2} + \frac{H}{2}\right)$$

$$P = \frac{10H^2}{4}\pi + 2H^2\sqrt{2}\,\pi$$

$$P = \frac{5H^2}{2}\pi + 2H^2\sqrt{2}\,\pi$$

$$V = \frac{\pi H}{3}\left(r_1^2 + r_1 r_2 + r_2^2\right)$$

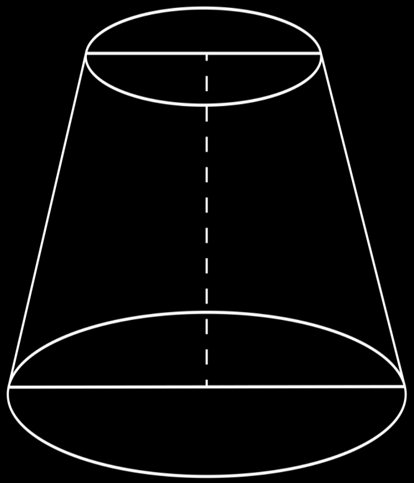
$$V = \frac{\pi H}{3}\left(\frac{H^2}{4} + \frac{9H^2}{4} + \frac{H}{2}\cdot\frac{3H}{2}\right)$$

$$V = \frac{\pi H}{3}\left(\frac{10H^2}{4} + \frac{3H^2}{4}\right)$$

$$V = \frac{\pi H}{3}\left(\frac{10H^2}{4} + \frac{3H^2}{4}\right)$$

$$V = \frac{\pi H}{3}\cdot\frac{13H^2}{4}$$

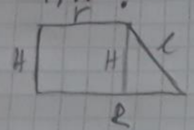
$$V = \frac{13H^3\pi}{12}$$

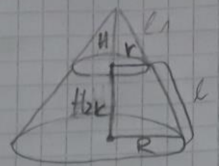


31.

R, r, l

$l_1, H = ?$





$$l^2 = H_{zk}^2 + (R-r)^2$$

$$H_{zk}^2 = l^2 - (R-r)^2$$

$$H_{zk} = \sqrt{l^2-(R-r)^2}$$

$$\frac{H}{R} = \frac{H_{zk}}{(R-r)}$$

$$H\cdot(R-r) = H_{zk}R$$

$$H = \frac{H_{zk}\cdot R}{(R-r)}$$

$$H = \frac{\sqrt{l^2(R-r)^2}\cdot R}{(R-r)}$$

$$l_1^2 = R^2 + H^2$$

$$l_1^2 = R^2 + \left(\frac{H_{zk}\cdot R}{R-r}\right)^2$$

$$l_1 = \sqrt{R + \left(\frac{H_{zk}\cdot R}{R-r}\right)^2}$$

$$l_1 = \sqrt{R + \left(\frac{\sqrt{l^2-(R-r)^2}\cdot R}{R-r}\right)^2}$$

$$l_1 = \sqrt{\frac{R}{1} + \frac{l^2-(R-r)^2\cdot R^2}{(R-r)^2}}$$

$$l_1 = \sqrt{\frac{R\cdot(R-r)^2 + l^2 - (R-r)^2\cdot R^2}{(R-r)^2}}$$

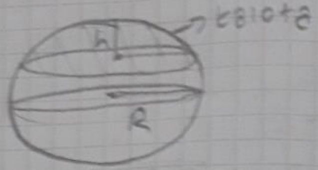
$$l_1 = \sqrt{\frac{Rl^2}{(R-r)^2}}$$

$$l_1 = \frac{Rl}{R-r}$$

28. Površina sferne kalote jednaka je $\frac{1}{3}$ površini lopte. Koji dio zapremine lopte pripada loptinom odsječku?

29. Odredi dio površine lopte, poluprečika 4, koji se vidi iz tačke koja se nalazi na rastojanju 8 od centra lopte.

28.



$P_k = \frac{1}{3} P_L$

$P_L = 4R^2\pi$

$P_k = 2R\pi h$

$V_L = \frac{4}{3} R^3 \pi$

$V_{odsečka} = \frac{\pi \cdot h^2}{3}(3R-h)$

$P_k = \frac{1}{3} P_L \Rightarrow 2R\pi h = \frac{1}{3} 4R^2\pi$

$\Rightarrow h = \frac{\frac{4}{3}R^2\pi}{2R\pi} = \frac{2}{3}R$

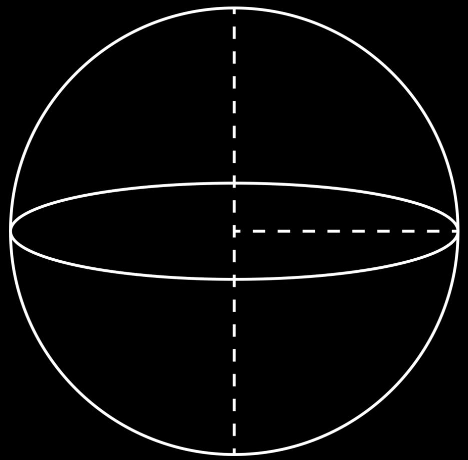
$V_{odsečka} = \frac{\pi \cdot h^2}{3}(3R-h)$

$= \frac{\frac{4}{9}R^2\pi(3R - \frac{2}{3}R)}{3}$

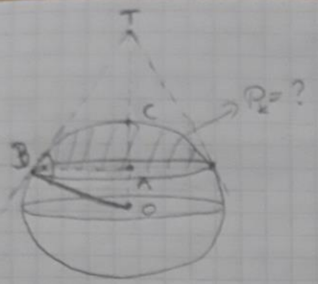
$= \frac{28\pi R^3}{81}$

$V_L = \frac{4\pi R^3}{3}$

$\frac{V_{odsečka}}{V_L} = \frac{\frac{28\pi R^3}{81}}{\frac{4\pi R^3}{3}} = \frac{\frac{28}{81}}{\frac{4}{3}} = \frac{84}{324} = \frac{7}{27}$



29.



$TO = 8$

$OC = OB = 4 = R$

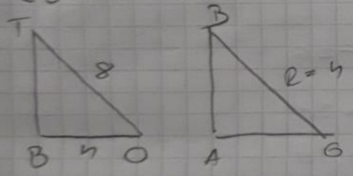
ΔBOT:

$OT^2 = TB^2 + OB^2$

$TB^2 = 8^2 - 4^2 = 48$

$TB = 4\sqrt{3}$

$\Delta BOT \sim \Delta ABO$



$TO : BO = BO : AO$

$8 : 4 = 4 : AO$

$AO = 2$

$h = AC = CO - AO = 4 - 2 = 2$

$h = 2$

$P_k = 2R\pi h$

$P_k = 2 \cdot 4 \cdot 3{,}14 \cdot 2$ $P_k = 16\pi$

~~$P_k \approx 50{,}24$~~